

Chaotic Dynamical Systems as Automata

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We discuss the replacement of discrete maps by automata, algorithms for the transformation of finite length digit strings into other finite length digit strings, and then discuss what it required in order to replace chaotic phase flows that are generated by ordinary differential equations by automata without introducing unknown and uncontrollable errors. That question arises naturally in the discretization of chaotic differential equations for the purpose of computation. We discuss as examples an autonomous and a periodically driven system, and a possible connection with cellular automata is also discussed. Qualitatively, our considerations are equivalent to asking when can the solution of a chaotic set of equations be regarded as a machine, or a model of a machine.

1. Introduction

It is remarkable that a stream of water flowing at high Reynolds number past a submerged object produces turbulence [1] with the greatest of ease while we can only in pale fashion and with great effort simulate such phenomena on a digital computer. There is a stark contrast here between turbulence and our success in simulating complex natural phenomena on an automaton. We can think of an automaton either as a machine or as a mathematical model of a machine. The machine or its model has internal states and can accept input in the form of digits or finite strings of digits $\varepsilon_i = 0, 1, 2, \dots, \mu - 1$ in some base $\mu = 2, 3, \dots$, of arithmetic, or merely symbols in an alphabet [2]. It produces output in the form of other finite length digit strings. According to von Neumann [3], any discretization of the Navier-Stokes equations or any other continuum system or set of differential equations for the purpose of digital computation is a replacement of the original system by an “artificial automaton”. One then studies the statistical properties of the artificial automaton’s output with the aim of gaining insight into the working of the original system.

It is one aim of this paper to discuss how a chaotic system [4, 5] of differential equations that defines a flow in phase space should be discretized

for computation in order to avoid introducing via truncation unknown and uncontrollable errors by the discretization process itself that can be magnified exponentially. The introduction of such errors eliminates any guarantee that the resulting artificial automaton will yield orbits whose statistical properties are similar to those of the original continuous system that we want to study, especially at long times. The importance of machine computation for the prediction of a chaotic system’s statistics certainly encourages us to think about the study of chaotic physical systems as computable maps, or machines, and while differential equations are useful for summarizing the geometric properties of systems in phase space, almost all analytic solutions that can be defined to exist are inaccessible for computation in a chaotic system. It is necessary to discover the computable properties [6] that may be hidden by the analytic language that was used in the formulation of the physics. For example, while it has been suggested that we should think of chaotic discrete maps as machines that produce a stream of numbers from a single definite input [5b], it has not been widely appreciated in physics that this is completely impossible for almost all maps and numbers that can be defined to exist. When we restrict to maps and numbers for which this is possible, we arrive directly at Turing’s computable numbers and functions. Likewise, it was also noted in [5b] that “solid links have yet to be forged between equations of motion and ‘chaotic solutions’”. In [6], a solid link was forged for the case of

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chaotic discrete maps. Those ideas are reviewed and explained in part 2 as a preparation for part 3, where we forge such a link for chaotic differential equations that define phase space flows. There, computable flows are replaced by computable maps which then can be replaced by automata, which means that the flow can be regarded as a model of a certain machine, or can be computed to within arbitrarily high precision by a machine, even though the machine's output is an a well-defined sense chaotic. In contrast, the result of an ordinary floating-point computation yields only periodic pseudo-orbits: the digit strings produced by fixed-precision computational methods are periodic rather than chaotic, even if the map that one attempts to compute is chaotic [6]. We believe that fluid turbulence falls into the category of deterministic chaos, although there is as yet no such solid link between Navier-Stokes equations and chaotic solutions. We discuss fluid turbulence only qualitatively and speculatively as a computation problem in the context of cellular automata near the end of the paper. In what follows, we replace computable flows by computable maps which can be regarded as machines, or automata. We discuss only qualitatively how automata may be replaced by cellular automata.

2. Discrete Maps as Automata

Consider first an iterated map on the unit interval

$$x_{n+1} = f(x_n, D), \quad (1)$$

where D is a control parameter and the iterates x_n belong to the continuum. If the map is chaotic (has a positive Liapunov exponent [4, 5]) then small changes in initial conditions will typically yield wildly different orbits. However, these orbits may generate well-defined statistics. If the map is mixing, for example, then almost all orbits are chaotic and an orbit starting from a single well-defined initial condition can generate a sequence of iterates whose distribution reflects a well-defined invariant density. That density can be used to discuss the consequences of any uncertainty in initial conditions, but is itself generated by a single typical orbit of the system. In this context, we mean by "chaotic" any non-periodic orbit of a map (1) with a positive Liapunov exponent [6]. Almost all initial conditions

and therefore almost all orbits are noncomputable. To stress this fact and to go on to identify deterministic chaos with randomness [7], which is non-computable, is unnecessary as well as misleading. While almost all numbers that can be defined to exist as elements of the continuum are algorithmically complex, hence noncomputable, there is a countable infinity of computable numbers and functions [8, 8b] and these can be used to formulate the computation of exact chaotic orbits of computable maps to any desired degree of precision [6]. The pitfall that must be avoided is the assumption that phase-space properties that are true "with measure one" will occur with certainty or even be easily approximated in computation. Since we do not have access to the continuum, but at best to a countable set of computable numbers generated by algorithms, analytic properties that are true with measure one do not appear with certainty in computation and are often difficult to generate, as is the case with normal numbers. This distinction between analytic and computable properties of maps has been emphasized elsewhere [6]. In what follows, we ignore algorithmically complex and other non-computable properties and restrict our discussion to computable orbits of chaotic maps. Otherwise, a chaotic map cannot be thought of as an automaton, or a machine.

While models of cellular automata [9] are typically written as integer maps, a digital representation is useful in order to view a computable map (1) of a continuous variable x explicitly as an automaton. From number theory, we know that expansions of numbers in any integral base μ ($= 2, 3, 4, \dots$) of arithmetic are complete [10] so that we can begin with the discretization, or digitalization of the map (1):

$$x_n = \sum_{i=1}^{N(n)} \varepsilon_i(n)/\mu^i, \quad D = \sum_{i=-M}^M \delta_i/\mu^i, \quad (2)$$

where M and $N(n)$ are integers, the "precisions" of D and x_n , respectively, and the expansion coefficients $\varepsilon_i(n)$ and δ_i can take on integral values $0, 1, 2, \dots, \mu - 1$. Digitalization in another context has been called "an application of information theory" and "a very clever trick for producing extreme precision out of poor precision" [3]. In the present context, the prospect of poor precision arises when x_0 is irrational, so that the digit string $\cdot \varepsilon_1(n) \varepsilon_2(n) \dots$ of x_n is non-periodic [10] and almost all such strings that can be defined to exist are noncomputable [8].

Extreme precision is possible when one has on hand an algorithm, an effective procedure, that generates the digit string. Such irrationals are called computable [8, 8b, 10b] and provide via symbolic dynamics a basis for understanding how the output of an automaton can be nonperiodic chaotic as opposed to merely unstable periodic. If f is a computable map and x_0 and D are computable numbers, then the digitalization (2) yields an algorithm whereby the string $\cdot \varepsilon_1(n+1) \varepsilon_2(n+1) \dots \varepsilon_{N(n+1)}(n+1)$ can be computed from finitely many of the digits $\varepsilon_i(n)$ and δ_i . Given that a finite number $N(n)$ of digits is held in the representation for x_n , the number $N(n+1)$ of digits that can be computed correctly in the string x_{n+1} can be estimated if the Liapunov exponent is known and the base of arithmetic μ is specified. An error in a single digit yields a completely different orbit of a chaotic map in the long run, and the number of iterations that can be performed before an error in the N -th digit of x_0 destroys the accuracy of the leading digit of x_n is $n \sim N \ln \mu / \lambda$. When f is a polynomial in x and D , then chaotic orbits may follow exactly from simple rational initial conditions. For example, with $f(x, D) = Dx(1-x)$, if we set $D=4$ and $x_0=1/8$, then we can choose $\mu=2$ to obtain the exact binary automaton $x_0=.001$, $f(x, 4)=100 * x * (1-x)$, $x_n = .\varepsilon_1(n) \varepsilon_2(n) \dots \varepsilon_{N(n)}$ which yields the exact chaotic output $x_1=.0111$, $x_2=.111111$, $x_3=.0000111111$, $x_4=.001110110001111111, \dots$. The orbit is non-periodic because the (2^n+2) -th bit in x_n is 1, followed by an infinite string of 0's. If instead we choose $x_0=.0011$, then we obtain an entirely different chaotic orbit due to the Liapunov exponent $\lambda = \ln 2$; after $n \sim 4$ iterations, even the leading bits of the two sequences will differ. Both here and above, we have used the digitalized form of the error propagation equation $\delta x_n \sim e^{n\lambda} \delta x_0$. In base μ arithmetic, that equation becomes $1 \sim e^{n\lambda} \cdot \mu^{-N}$, which yields the above mentioned relaxation time for machine error propagation, $n \sim N \ln \mu / \lambda$.

If we ask what is the length of the logistic map's digit string as n increases, it is $\sim 2^n$ when $D=4$ and x_0 is rational, so that the "error-magnification property", which is the essence of deterministic chaos, is the direct cause of the requirement of increasing precision per iteration of the map, because the effect of $\lambda > 0$ is to cause a finite average rate of information flow from right to left in the digit

strings as n increases. Periodic orbits of this map are dense in the unit interval, but are not easily computed by forward iteration of the logistic map since those orbits are unstable and typically begin from irrational initial conditions. However, unstable periodic orbits can be located and then computed if we first transform to the tent map [4] where the unstable periodic orbits begin from rational initial conditions. This transformation permits unstable periodic orbits of the logistic map to be computed to any desired finite precision N as a uniform approximation to the exact orbits, but the chaotic orbits follow directly by iteration of the map from simple rational initial conditions. In the absence of a transformation to the tent map, unstable periodic orbits can still be computed by solving the defining polynomial equations by Newton's method.

Given a computable map f with computable numbers x_0 and D , one can in principle compute uniform approximations to *chaotic* orbits generated by *irrational* initial conditions x_0 . If f is defined by a non-terminating infinite series, we can write $f(x, D) = S_N(x, D) + R_N(x, D)$ where $S_N(x, D)$ is an N -th order polynomial approximation to f with remainder R_N . The size of $R_N(x_0, D)$ yields an estimate for the number of iterations that can be computed before the error committed by ignoring R_N destroys $\varepsilon_1(n)$, namely, $n \sim -\ln |R_N(x_0, D)| / \lambda$. The violation of this condition will yield a pseudo-orbit, even if $S_N(x, D)$ should be evaluated exactly. In that case, we can regard $R_N(x, D)$ as a non-uniform error per iteration due to the fact that the computed map S_N differs from the exact map f . If enough terms are held in $S_N(x, D)$ then uniform approximations to exact chaotic orbits can be computed for a finite number n of iterations. The main goal of this paper is to point out how this can be extended to the computation of chaotic phase space flows.

In contrast, pseudo-orbits are computed orbits that are not true orbits of the dynamical system, and are typically a consequence of the use of floating-point arithmetic. In a pseudo-orbit computation, it is sometimes possible to interpret the results to mean that a section of an orbit is computed accurately only for a very short while from an exact initial condition, but that error magnification rapidly causes the computed orbit to diverge from the exact orbit in an iteration time $n \sim N \ln \mu / \lambda$, where N is the number of digits carried accurately by the

machine (in single precision floating point binary arithmetic $N < 30$, e.g.). In such a case, the pseudo-orbit is made up of a sequence of short, early-time segments of a large number of different approximations to different exact orbits that are patched together piecewise in a time sequence, and the pseudo-orbit is periodic. For dissipative systems with a rapid volume contraction rate, it is sometimes argued that such pseudo-orbit calculations rapidly contract toward a strange attractor, while for conservative systems that are mixing it is typically thought that the distribution of these segments reflects the system's invariant density. That interpretation of the output may be possible at some minimum length scale that is much larger than the scale set by the machine's precision, but it is by no means guaranteed [6, 12]. Furthermore, since pseudo-orbits are typically periodic, they are more likely to reflect the statistics of unstable periodic orbits rather than those of nonperiodic (chaotic) ones [12b], whereas it is long time behavior of a *single* generic non-periodic orbit whose closure is the attractor. Whether a given unstable periodic orbit correctly reflects the statistics of a chaotic attractor at some coarse-grained level of description is an unsolved problem, except for linear shift maps [12b]. Whether or not a *computable* initial condition can be found that generates a "generic" chaotic orbit is an entirely different question. Presumably, the iterates of the orbit starting from $x_0 = 1/8$ generate a closure that is the entire interval as $n \rightarrow \infty$, and these iterates should be distributed according to the invariant density

$$p(x) = \{\pi^2 x(1-x)\}^{-1/2}.$$

The generation of chaotic digit patterns from "simple seeds" occurs also in certain simple cellular automata [9, 11].

The digitalization (2) of discrete maps (1) leads naturally to a coarse-graining of the unit interval or, more generally, the phase space into μ^N bins, each of width μ^{-N} , where N is the precision to within which the iterates $x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \dots \rightarrow x_n$ of orbit are computed. A computation is equivalent to dropping balls into these bins with a definite time sequence, and from the bin statistics that follow from a single orbit, one can compute all statistical properties of interest, such as correlation functions, fractal dimensions, power spectra, etc. From the point of view of computation, the coarse-graining of

the phase space into bins whose sizes are determined by the precision is fundamental since N is finite in any computation, and it is the restriction to computable numbers that limits us at best to a definite, countable subset of all possible numbers that can be defined to exist in the continuum. If instead of the real arithmetic with the increasing precision per iteration that is demanded by the action of a positive Liapunov exponent one uses either floating-point arithmetic or fixed-precision real arithmetic [13], then pseudo-orbits will typically follow from a computation [6, 12, 12b]. Pseudo-orbits are discussed within the context of analysis in Lanford [14], while the difficulty of knowing the errors introduced into any computation by the use of floating-point arithmetic was mentioned by Knuth [13]. Pseudo-orbits in the context of machine computation are discussed in [12b].

3. Phase Flows Generated by Differential Equations as Discrete Maps

The idea in what follows is to make a definite connection between chaotic ordinary differential equations and chaotic solutions in the form of computable maps; e.g., Taylor expansions are computable, and this provides one such means for linking these ideas. In [14b], it was suggested to regard computers as a certain class of differential equation, but no attention was paid to computability in that discussion. The computable properties, however, are exactly the machine-like ("automaton") properties that we need in order to forge such a link.

There is no essential difference between numbers and functions in computability theory [10b]. We should on general grounds expect that the arbitrary discretization of a set of differential equations for computation as an iterated map will produce unknown and uncontrollable errors when the system is chaotic. To be explicit, we have in mind the replacement of chaotic differential equations that define a phase-flow by a set of iterative equations of fixed order, e.g., a fifth order Runge-Kutta method. In that case, one doesn't know a priori that the statistical properties of an artificial automaton that are obtained by truncation reflect accurately the statistical properties of the original system when pseudo-orbits are computed for times that are long

compared with the time scale over which the fifth order approximation is valid.

In order to be explicit, consider the Lorenz model

$$\dot{x} = \sigma(y - x), \quad \dot{y} = (r - z)x - y, \quad \dot{z} = xy - bz, \quad (3)$$

which is chaotic if $r \geq \sigma(\sigma + b + 3)/(\sigma - b - 1)$ [4, 5]. The exact solution for an orbit can be formally written down as a power series in the time, and if we write $x(t) = x_n$, $y(t) = y_n$, $z(t) = z_n$, $x(t + \tau) = x_{n+1}$, etc. where τ is small enough that the series converges but is otherwise arbitrary, then the solution for the orbit can be written as a discrete map, the stroboscopic map,

$$\begin{aligned} x_{n+1} &= x_n + \tau \sigma(y_n - x_n) + \tau^2 \sigma(\sigma(y_n - x_n) \\ &\quad - (r - z_n)x_n - y_n)/2 + \dots, \\ y_{n+1} &= y_n + \tau((r - z_n)x_n - y_n) + \tau^2((r - z_n)\sigma(y_n - x_n) \\ &\quad - x_n(x_n y_n - bz_n - (r - z_n)x_n - y_n)/2 + \dots, \\ z_{n+1} &= z_n + \tau(x_n y_n - bz_n) + \tau^2(\sigma y_n(y_n - x_n) \\ &\quad + x_n((r - z_n)x_n - y_n) - b(x_n y_n - bz_n))/2 + \dots, \end{aligned} \quad (4)$$

which is just the effect of the time evolution operator U_t as an infinite order differential operator. The existence of U_t for all t defines a flow in phase space [15], and the group property $U_{n\tau} = U_\tau^n$ is reflected in the fact that

$$\begin{aligned} x_n &= x_0 + n\tau\sigma(y_0 - x_0) + (n\tau)^2\sigma(\sigma(y_0 - x_0) \\ &\quad - (r - z_0)x_0 - y_0)/2 + \dots, \\ y_n &= y_0 + \dots, \\ z_n &= z_0 + \dots, \end{aligned} \quad (5)$$

making iteration of the map (4) unnecessary. It is no surprise that such a map can be written down for small enough τ , but it has not been previously emphasized that this map can be seen as the link between differential equations and automata. One can define the map by (x_n, y_n, z_n) as given, e.g., by (5) and all its analytic continuations. Given a convergent power series, each term of which is a computable function of computable parameters and initial conditions, we have on hand a computable function. However, in order to compute flows from (5) without making uncontrollable errors, it is necessary to know how many terms must be held in a polynomial approximation to (5) for a given number of iterations n . However, while the above definition is analytically and formally appealing, from the point of view of *construction* of digit strings, it is necessary instead to use Taylor expan-

sions with finite remainder terms,

$$\begin{aligned} x_n &= S_N(x_0, y_0, z_0, n) + R_N(x_0, y_0, z_0, n), \\ y_n &= \dots \\ z_n &= \dots, \end{aligned} \quad (5b)$$

where S_N is an N -th order polynomial, in order to be able to compute and thereby control the error. If we do not *compute* a decimal expansion for R_N , then there is no way to avoid uncontrollable truncation errors in the iterates (x_n, y_n, z_n) . The problem here is not more difficult than that of computing the orbit of a discrete map (1) given as three coupled infinite series. Although, we have discussed Taylor expansions generated directly by the differential equations, there is no reason why our argument should not apply equally well to the construction of the flow's map by approximating it by other sets of functions. In any case, uncontrollable errors cannot be ruled out unless reliable error estimates are computed.

If instead we should arbitrarily truncate the series to fourth order, e.g., then we commit an error after a time $t = n\tau \sim |f^{(4)}/f^{(5)}|$ where $f^{(m)}$ is the order of magnitude of the m -th order coefficients of τ^m in (5). The resulting error will be magnified exponentially by the action of a positive Liapunov exponent, and even if we should evaluate the arithmetic of the 4th order map exactly (thus avoiding a pseudo-orbit of that map), we shall have replaced the Lorenz model by an artificial automaton, the 4th order map, whose statistical properties may differ from those generated by an exact orbit of (5), especially for long times t and large values of the precision N . The latter are exactly the limits that one needs in order to compute fractal dimensions. We turn now to a discussion of a periodically driven system along with questions of its modelling by simpler maps.

In the case of periodically driven systems the time-advance operations do not form a one parameter group except for discrete time steps $n\tau$, where τ is the period of the driving force [15]. As an example, consider the Josephson Junction equation [16]

$$\ddot{\theta} + \beta\dot{\theta} + \gamma\sin\theta = A + B\sin\omega t. \quad (6)$$

If we write $r = \dot{\theta}$ then we can consider orbits in the two-dimensional (θ, r) phase space:

$$\begin{aligned} \theta(t + \tau) &= \theta(t) + \tau\dot{\theta}(t) + \tau^2\ddot{\theta}(t)/2 + \dots, \\ r(t + \tau) &= r(t) + \tau\dot{r}(t) + \tau^2\ddot{r}(t)/2 + \dots. \end{aligned} \quad (6a)$$

If then we write $\theta_n = \theta(t + n\tau)$ and $r_n = r(t + n\tau)$ then the complete Taylor expansion yields a time advance map of the form

$$\begin{aligned}\theta_{n+1} &= G_1(\theta_n, r_n), \\ r_{n+1} &= G_2(\theta_n, r_n),\end{aligned}\quad (6b)$$

where G_1 and G_2 have no explicit n -dependence only if we choose $\tau = 2\pi/\omega$ [15]. As before, this map is called the stroboscopic map [16] and the functions G_1 and G_2 can be constructed as formal power series (6a) by repeated differentiation of the differential equation (6). If we compute the first three terms in the Taylor expansion (6a), we obtain

$$\begin{aligned}\theta_{n+1} &= \theta_n + (A\tau^2/2 - \beta A\tau^3/3! + \omega B\tau^3/3!) \\ &\quad - (\gamma\tau^2/2 - \beta\gamma\tau^3/3!) \sin \theta_n - \gamma\tau^3 r_n \cos \theta_n/3! \\ &\quad + (\tau - \beta\tau^2/2 + \beta^2\tau^3/3!) r_n + \dots, \\ r_{n+1} &= r_n (1 - \beta\tau + \beta^2\tau^2/2) \\ &\quad + (A\tau - A\beta\tau^2/2 + B\omega\tau^2/2) \\ &\quad - (\gamma\tau + \gamma\tau^2\beta/2) \sin \theta_n + \\ &\quad + \gamma\tau^2 r_n \cos \theta_n/2 + \dots.\end{aligned}\quad (6c)$$

Since these series are expected to converge at high frequencies (small τ), we can argue that by inspection of the general n -th order terms in these power series, the stroboscopic map can be summarized into the general form

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega + h(\theta_n, r_n) + (1 - J) r_n / \beta, \\ r_{n+1} &= J r_n + \Omega' + g(\theta_n, r_n)\end{aligned}\quad (6d)$$

for all times $\tau = 2\pi/\omega$, where h and g are periodic in θ_n with period 2π , Ω and Ω' are constants and $J = e^{-\beta\tau}$ is the Jacobian of (6b).

In the recent literature [16] the dissipative sine-standard map

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega - K \sin \theta_n + J r_n, \\ r_{n+1} &= J r_n - K \sin \theta_n\end{aligned}\quad (6e)$$

has been studied in an attempt to model (6b). This modelling has rested upon the hope for universal behavior of (6b) and (6e), and with that notion in mind, we can ask whether there is some range of parameters where (6e) should approximate (6d) and hence should produce similar bin statistics in computation. If we go to dimensionless variables $\tilde{t} = t/\tau$, $\tilde{r}_n = \tau r_n/2$, $\tilde{\beta} = \beta\tau$, and $\tilde{\gamma} = \gamma\tau^2$, we can view

(6c) as an expansion in small parameters $\tilde{\beta}$ and $\tilde{\gamma}$, which restricts us to the subcritical regime when only a few terms are held in (6c). To lowest non-vanishing order in $\tilde{\beta}$ and $\tilde{\gamma}$, (6c) yields the approximate map

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega - \tilde{\gamma}(1 - \tilde{\beta}/3) \sin \theta_n/2 \\ &\quad - (1 - \tilde{\beta}/2) \tilde{r}_n - \tilde{\gamma} \tilde{r}_n \cos \theta_n/3,\end{aligned}\quad (6f)$$

$$\tilde{r}_{n+1} = (1 - \beta) \tilde{r}_n + \Omega' - \tilde{\gamma}(1 - \tilde{\beta}/2) \sin \theta_n/2 + \tilde{\gamma} \tilde{r}_n \cos \theta_n,$$

which does not reduce to the sine standard map due to the presence of the cross-terms $\tilde{r}_n \cos \theta_n$. If (6e) should approximate (6d) for parameter values that are near the critical line, then we infer from other work [18] that a singular perturbation expansion may be necessary in order to analyze the question. Slightly below and at criticality, numerical experiments have been used to argue that both (6b) and (6e) should be well-approximated by a one-dimensional circle map, thereby exhibiting universal critical behavior. However, there is no reason to believe that (6b) and (6e) exhibit universal statistics beyond the critical line and within the chaotic regime.

The use of universality arguments to solve for the properties of (6d) has succeeded only in the restricted case where the r_n dependence is ignored, yielding a one dimensional circle map. If we choose parameters in (6d) so that the motion is regular (stable periodic or quasi-periodic), then contraction of phase-volume elements should yield either points or closed curves as attractors as $n \rightarrow \infty$, suggesting a one-dimensional map in that limit. If we assume without proof that r_n is a “fast” variable and θ_n a slow one, then (6d) should reduce asymptotically to a one-dimensional circle map

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Omega + H(\theta_n), \\ H(\theta_n + 2\pi) &= H(\theta_n) \bmod 2\pi.\end{aligned}\quad (7)$$

In the stable regime, there is numerical evidence for this reduction in dimension of the map [16], although analytic evidence has been obtained only under restrictions characteristic of the outer solution in a singular perturbation expansion, and where the right-hand side of (6) was replaced by a periodic series of delta functions [18]. In the chaotic regime where fractal attractors can occur, there is no good reason to assume an integral reduction in dimension of the map (6d).

For circle maps (7), the quasi-periodic motion in the regular regime is universally equivalent to a

rotation [19], and that universality was investigated numerically at the critical point, where (7) becomes non-invertible, by Shenker [20] under the restriction that the irrational winding numbers of the maps have the same asymptotic entries in their continued fraction expansions. The idea of restricted universality near critically for one-dimensional circle maps with the same winding numbers has been extended [17, 21], but it is known that the assumption of a one-dimensional circle map contradicts (6b) beyond criticality [16, 17].

Our viewpoint emphasizes that the fundamental limitation on the prediction of the properties of a chaotic system is the growth in demand for more digits, or computer time [6], which is illustrated in the logistic map at $D = 4$ by the growth in the string of length $\sim 2^n$ per iteration, starting from a simple rational initial condition, combined with a finite rate of information flow from right to left in the digit strings. That is, all of the 2^n digits are significant within a “relaxation time” $n \sim N \ln \mu / \lambda$. In computation, this growth in digit string length is equivalent to the uncertainty in initial conditions that is discussed in the context of analysis. From the point of view of computation, the exponential growth rate of digit strings is the fundamental property that characterizes deterministic chaos. Since when we pay attention to the demand for this increased precision, the demand for computer time becomes the factor that limits the prediction of the future from the past in these systems, it is interesting to ask whether there is some encoding scheme whereby the automaton can be replaced by a simpler dynamical system for the purpose of computation, namely, a cellular automaton. In what follows, we follow a speculative and incomplete line of thought that leads from automata to cellular automata.

4. Dynamical Systems Encoded as Universal Cellular Automata

Our discussion of dynamical systems as automata has led us quite naturally to consider the question whether it may be useful to try to study dynamical systems as cellular automata, in contrast with the automata discussed above which were not encoded in cellular form. The main idea of a cellular automaton is to construct either physically [22, 23]

or as a mathematical model [9] an array of locally interacting identical processors. Each processor is an automaton, receiving finite strings of digits as input and providing a new string of digits as output. It is easy to make mathematical models of cellular automata. For example, the binary map

$$\varepsilon_j(n+1) = \varepsilon_{j+1}(n) + \varepsilon_{j-1}(n) \bmod 2 \quad (8)$$

is such a model where $\varepsilon_j(n) = 0$ or 1 , j is a discrete space variable and n is a discrete time variable. Qualitatively, one imagines a long one-dimensional array of processors (cells) coupled with nearest-neighbors, so that the j -th processor receives as input $\varepsilon_{j+1}(n)$ and $\varepsilon_{j-1}(n)$ at time n , and updates at $n+1$ according to rule (8). This is the idea of parallel processing [22] since all cells update simultaneously rather than sequentially.

Certain two-dimensional cellular automata are said to be capable of universal computation, which presumably means that any computable function [2, 8b] should in principle be programmable as an acceptable initial condition for the automaton. The computation is then carried out by the dynamics of the cellular automaton, in analogy with (8). For example, Conway’s game of life is known to be capable of universal computation [9], so that one might expect to be able to use it to compute, e.g., the logistic map. But while it is easy to rewrite the logistic map as a binary automaton, in order to encode that binary information as an initial condition for the game of life or for any other cellular automaton capable of universal computation, it is first necessary to construct a “machine language”, or encoding scheme for that particular automaton. If one does not have on hand the necessary encoding scheme, there is no way to use the cellular automaton as a computer, and its universality stands only as an existence theorem in mathematics.

If we should have on hand an appropriate machine language for, say, Conway’s Game of Life, then we might follow the following line of reasoning. Dissipative chaotic systems, one dimensional nonlinear chaotic maps, and simple cellular automata are capable of generating chaotic digit strings from “simple seeds”, or rational initial conditions. The latter has been emphasized by Wolfram, who also has suggested that our understanding of turbulence in fluids may be improved by the study of cellular automata [11]. In general, we see no reason why, e.g., the Lorenz model, as well, should not

produce non-periodic orbits from rational initial conditions and rational control parameters, since we have already seen now this works in the logistic map. If we should consider the logistic map as a binary automaton with a simple three bit initial condition (cf. Sect. 2 above), then we can perhaps imagine encoding that information as an initial condition for a two dimensional cellular automaton that is capable of universal computation. In the latter case, the dynamics will be simple, but the initial condition, or program, will not likely be simple: we expect that the complexity of the two systems will be more or less the same, no matter which method of computation one chooses. Here, it seems to be a trade of complicated dynamics (multiplication of long binary strings) with simple seeds for simple dynamics with a complicated seed. Whether anything worthwhile can be learned from such an approach is unclear, since the program has not been carried out, but we suspect that universal cellular automata are too general to be of much use as mathematical models for understanding the physics of fluid turbulence.

5. Summary

By paying attention to the requirement of increasing precision per iteration that is required by a chaotic map, uniform approximations to chaotic orbits of computable maps can be computed at finite times with controlled precision. That is because computable maps can be rewritten as automata in some definite base of arithmetic. The map is then equivalent to a set of rules for transforming one digit string into another. These digit strings may in principle be infinite, and computable maps yield algorithms that permit these digit strings to be computed to any desired finite length. The inability to predict the future from the past in these systems at arbitrarily long times follows from the fact that (a) you are not allowed to make an error in a single digit, and (b) positive Liapunov exponents produce increasingly long digit strings that must be computed in order to compute an orbit, making computation time the most fundamental limitation.

Flows generated by differential equations can be written as a discrete map, the stroboscopic map, via finite Taylor expansions, e.g. For periodically driven systems the stroboscopic map is unique, while for

autonomous systems there is no unique integration time step, hence no unique map. In either case, computable maps give rise to computable chaotic orbits. To know one of these orbits even to low precision, computations must be carried out to high precision. Correct attractor statistics should follow from the correct computation of a single typical nonperiodic orbit, but to know even the first digit in a single coordinate after a large number of iterations, one must know the previous iterations to within a very high accuracy. The main problem here is the estimate of errors made in approximating the map by a polynomial.

Cellular automata provide simple integer map models of chaotic behavior [9] but to program a discretized dynamical system on a universal cellular automaton requires knowledge of a “machine language” for that automaton. Given a machine language and the correct map, the flow of a complicated dynamical system could then in principle be written as a complicated initial condition for a universal cellular automaton. There seems, a priori, no reason to expect that such computer programs for universal cellular automata should be simpler than the original automata that follow directly from digitalization of maps.

Recently, the Navier-Stokes equations have been modelled approximately by a lattice gas [24]. That approximation is uncontrolled and simulates certain features of the Navier-Stokes equations while violating others. It will be interesting to find out whether a computer architecture based upon that model [24] will lead to any improvement in our ability to understand turbulent flows.

Considerations of automata help to emphasize the difference between nature and our attempts to model nature, a difference that is easily forgotten in the formulation of nonlinear continuum field theories. And when we face squarely the limitations of finite precision, with the help of Turing’s ideas, then lattice gas and other integer map models may be viewed as more, not less fundamental, especially so far as our ability to predict, which means to compute, is concerned.

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